Point form of Circle and Ellipse

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This Article is from a published paper named **Formulating Tangent Loci** (ICCIT'99 proceedings page 217-221).
The idea of point circle and point ellipse and some of their applications.

Part 1
Point Circle

1. **Point circle:**

Point circle is a circle of radius zero. We know circle having center at (h,k) and radius r is represented by equation:

$$(x-h)^2+(y-k)^2=r^2 \qquad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots(1.1)$$

If r = 0 then equation (1.1) reduces to

$$(x-h)^2+(y-k)^2=0 \qquad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots(1.2)$$

Equation (1.2) is a point circle at point (h,k).

The point (h,k) which is satisfied by the above equation is not just a point rather it is the contraction of the whole circle. The collapse of all the points to the center by the radius being zero makes a point circle. This understanding helps to find some applications of point circle in solving various types of problems related to tangency.

Consider the figure below.

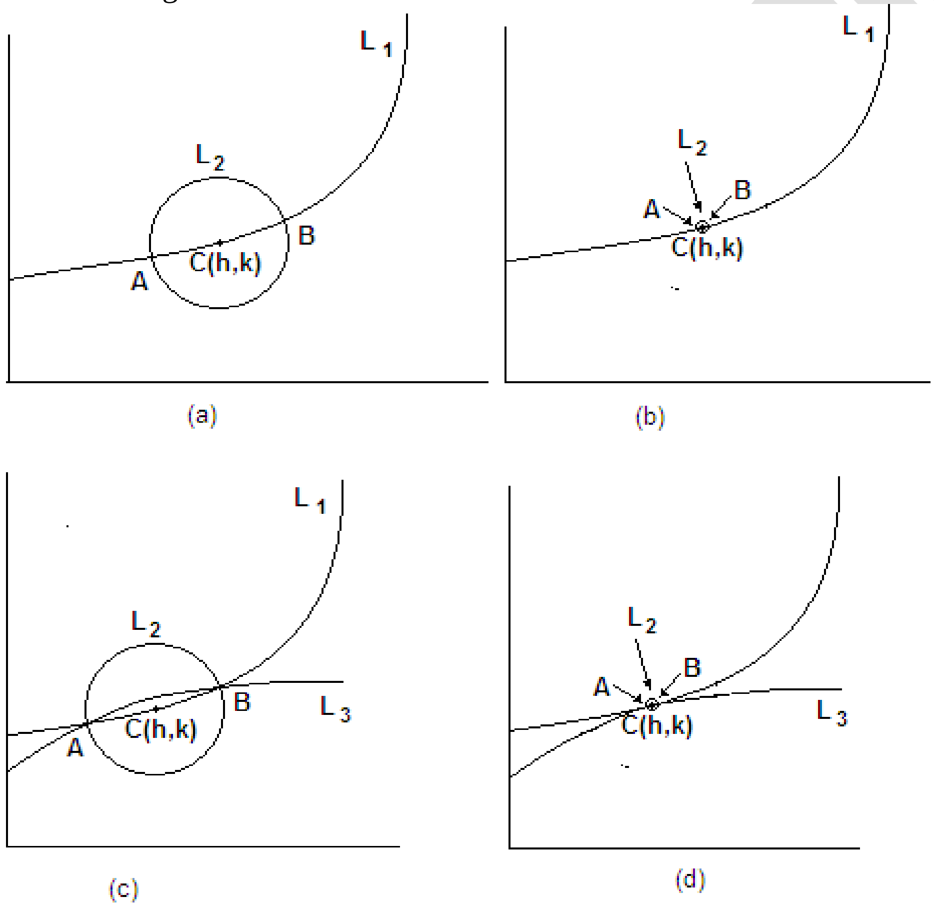

Fig: 1.1

In figure (a) the circle L_2 having center (h,k) on the curve L_1. If the circle reduces to a point circle then all the points on it collapse on the point (h,k) on the curve L_1[fig (b)].

Let L_3 be a curve passing through the point of intersection of L_1 and L_2 [fig(c)]. When circle L_2 reduces to point circle, the points of intersection A and B coincides to make the curve L_3 a tangent curve to the point (h,k) of L_1 [fig (d)].

Theorem:
Equation of the family of touching circles [C_2] touching a circle C_1 at point (h,k) is given by

$$C_2 \equiv C_1 + \lambda\{(x-h)^2+(y-k)^2\}=0 \; ; \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

[λ is an arbitrary constant]

Special case 1:
Let C_1 [$C_1 \equiv x^2+y^2+2gx+2fy+c=0$] be the equation of a circle and (h,k) is a point on it. For λ= -1 equation (1.3) reduces to

$$x^2+y^2+2gx+2fy+c-\{(x-h)^2+(y-k)^2\}=0 \ldots \quad \ldots \quad \ldots \quad \ldots(1.4)$$

It is observable that square terms are canceled out from equation (1.4) so it is nothing but the equation of the tangent at point (h,k) on the given circle C_1.

So we may simplify as
Circle - Point Circle ≡ Tangent line on the Circle

- **Example 1.1:**

Find the equation of tangent at point (0,2) on the circle $x^2+y^2+14x-6y+8=0$

Solution:
Equation of point circle at point (0,2) is given by

$(x-0)^2+(y-2)^2=0$

$\Rightarrow x^2+y^2-4y+4=0$

So the equation of the tangent

$x^2+y^2+14x-6y+8-(x^2+y^2-4y+4)=0$

$\Rightarrow 14x-2y+4=0$

$\Rightarrow 7x-y+2=0$ Ans.

Now verify the result using traditional formula[1]

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

$\Rightarrow x\times 0+y\times 2+7(x+0)-3(y+2)+8=0$

$\Rightarrow 7x-y+2=0$ Ans.

[1] [Application of point circle is not giving any favor to reduce the difficulty here. But as an application it should be noticed.]

- **Example 1.2:**

Find the equation of the tangent to the circle $x^2 + y^2 - 6x + 4y - 3 = 0$ from the point $(-7,2)$.

Solution:
Let the equation of the tangent

$$x^2 + y^2 - 6x + 4y - 3 - \{(x-h)^2 + (y-k)^2\} = 0$$

$$\Rightarrow x(2h-6) + y(2k+4) + h^2 + k^2 - 3 = 0 \quad \ldots.. \quad \ldots.. \quad \ldots..(1.5)$$

(h,k) Lies on the circle and the point $(-7,2)$ lies on the tangent. So,

$$h^2 + k^2 - 6h + 4k - 3 = 0 \quad \ldots.. \quad \ldots.. \quad \ldots..(1.6)$$

And from (1.5) we obtain

$$-7(2h-6) + 2(2k+4) + 49 + 2 - 3 = 0$$

$$\Rightarrow -14h + 4k + 98 = 0 \quad \ldots.. \quad \ldots.. \quad \ldots.. \quad \ldots..(1.7)$$

Now the values of (h,k) can be calculated from (1.7) and (1.6).
Here we will have two values for both *h* and *k* to have two required tangents from the given point.

Above example is another application of point circle but not a good one to reduce any difficulty from traditional method.

Special case 2:

We can easily deduce another theorem to obtain the equation of the circle touching a straight line passing through a given point.

Let $L_{1\equiv}$ $ax + by + c = 0$ is a straight line and (α, β) be any given point on it. Now for the geometry of [curve]-[point circle] intersection as we described before by fig.(1.1) the equation of the circle touching the straight line through point (α, β) is given by,

$$ax + by + c + k\{(x-\alpha)^2 + (y-\beta)^2\} = 0 \quad [k \text{ is an arbitrary constant}]$$

For $k \neq 0$ and assuming $\lambda = \dfrac{1}{k}$ we can re-write the above equation as follows

$$(x-\alpha)^2 + (y-\beta)^2 + \lambda(ax + by + c) = 0 \quad [\lambda \text{ is an arbitrary constant}]$$

Theorem:

Equation of family of circles touching a straight line $ax + by + c = 0$ through point (α, β) is given by,

$$(x-\alpha)^2 + (y-\beta)^2 + \lambda(ax + by + c) = 0 \quad \ldots.. \quad \ldots.. \quad \ldots..(1.8)$$

$$[\lambda \text{ is an arbitrary constant}]$$

- **Example 1.3:**

Find the equation of the circle touching the straight line $2x-3y-1=0$ at point $(2,1)$ and passing through a point (4,3).

Solution:

Let the equation of the circle be,

$$(x-2)^2+(y-1)^2+\lambda(2x-3y-1)=0$$

as the point (4,3) lies on circle,

$$(4-2)^2+(3-1)^2+\lambda(8-9-1)=0$$
$$\Rightarrow 8-2\lambda=0$$
$$\Rightarrow \lambda=4$$

So putting the value of λ in the assumed equation we have the equation of the circle

$$(x-2)^2+(y-1)^2+4(2x-3y-1)=0$$
$$\Rightarrow x^2+y^2+4x-14y+1=0 \text{ Ans.}$$

In the above solution we got remarkable advantage for using the point circle application. The traditional method cannot avoid assuming of $(g, f.c)$ at least three arbitrary constants whereas using point circle we solve it as a problem of only one arbitrary constant λ.

- **Example 1.4:**

Find the equation of circle touching the circle $x^2+y^2+6x-8y-11=0$ at point (3,4) and passing through the point (7,0).

Solution:

Let the equation of the circle be

$$x^2+y^2+6x-8y-11+\lambda\{(x-3)^2+(y-4)^2\}=0 \ldots\ldots \quad \ldots\ldots \quad \ldots.(1.9)$$

For it is passing through the point (7,0)

$$7^2+0^2+6\times7-8\times0-11+\lambda\{(7-3)^2+(0-4)^2\}=0$$
$$\Rightarrow 80+32\lambda=0$$
$$\Rightarrow \lambda=-\frac{5}{2}$$

Put the value of λ in (1.9) to find the circle.

- **Example 1.5**

Find the equation of the circle touching the straight line $5x-3y-2=0$ at point (1,1) and of radius 4.

Solution:

Assume the equation of the circle

$$(x-1)^2+(y-1)^2+\lambda(5x-3y-2)=0 \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots(1.10)$$

(Radius)^2 = $\left(\frac{-2+5\lambda}{2}\right)^2+\left(\frac{-2-3\lambda}{2}\right)^2-(2-2\lambda)=16$

$$\Rightarrow 34\lambda^2=64$$

$$\Rightarrow \lambda=\pm\frac{4\sqrt{2}}{\sqrt{17}}$$

Put the value of λ in (1.10) to obtain the equation of the circle.

- **Example 1.6:**

Find the equation of the circle which is passing through origin and touching the straight line $7x-3y-1=0$ at point (1,2).

Solution:

Let the equation of the circle be

$$(x-1)^2+(y-2)^2+\lambda(7x-3y-1)=0 \quad \ldots.. \quad \ldots.. \quad \ldots.. \quad \ldots..(1.11)$$

For passing through origin the constant term of (1.11) = 0,

$$\Rightarrow 1+4-\lambda=0$$

$$\Rightarrow \lambda=5$$

Putting the value of λ in (1.11) we obtain the equation of the circle

$x^2+y^2+33x-19y=0$. Ans.

We here discussed the application of point circle in two dimensional cases. Similarly we can use the idea of **Point Ellipsoid** [$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=0$] for the cases of three dimensions as well. Equation of surface touching another surface at a given point can be obtained using point ellipsoid.